

Incompressible Stars and Fractional Derivatives

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Fractional calculus is an effective tool in incorporating the effects of non-locality and memory into physical models. In this regard, successful applications exist ranging from signal processing to anomalous diffusion and quantum mechanics. In this paper we investigate the fractional versions of the stellar structure equations for non radiating spherical objects. Using incompressible fluids as a comparison, we develop models for constant density Newtonian objects with fractional mass distributions or stress conditions. To better understand the fractional effects, we discuss effective values for the density, gravitational field and equation of state. The fractional objects are smaller and less massive than integer models. The fractional parameters are related to a polytropic index for the models considered.

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I. INTRODUCTION

Fractional calculus offers a convenient way to introduce memory, non locality and other fractional effects into physical models that are not covered by standard non-linear but local models [1, 2]. Fractional generalizations of some of the basic differential equations of physics have led to new understandings of the dynamics underlying macroscopic phenomena in a wide range of areas [3–5] like anomalous diffusion, signal processing and quantum mechanics [6]. Applications to stellar structure include the work of El-Nabulsi [7], who considered a fractional equation of state for white dwarf stars. Two of the fundamental tools in building models for compact objects are the equation of hydrostatic equilibrium (HSE) and the spherical symmetry condition (mass-density relation). In this paper we develop some new models for Newtonian incompressible, static and spherically symmetric stars by

fractionalizing these two conditions. Constant density, incompressible fluids are useful in modeling because they provide analytic solutions that can be compared to the behavior of real physical systems [8, 9]. Solid planets, white dwarfs and neutron stars may be approximated by incompressible fluid models [9, 10]. Recent work has linked the Navier-Stokes equations for an incompressible fluid to solutions of the Einstein field equations in higher dimensions [11–13]. Incompressible fluids have also been linked to other model fluid descriptions [14]. Fractionalizing the structure equations for a Newtonian constant density star provides some new stellar models with interesting features not found in constant density stars described by the integer structure equations.

Because of high densities, some stellar models use general relativity to incorporate curvature effects. Naturally, the general hydrostatic equilibrium equation for Newtonian fluids follow from the zero covariant divergence of the stress energy tensor, $T^{ij}_{;j} = 0$. Fractionalizing partial derivatives is usually just a replacement of an integer derivative with a fractional derivative. Fractionalizing covariant derivatives in general relativity is more complex than fractionalizing a partial derivative because of the contributions of Christoffel connections to the covariant derivative and to the structure of the tensor definitions. Spherically symmetric relativistic and non-radiating stellar models usually start with the metric [15]

$$ds^2 = -e^{\nu(r)}c^2dt^2 + (1 - 2Gm(r)/c^2r)^{-1}dr^2 + r^2d\Omega^2, \quad (1)$$

and the Einstein field equations are solved for $\nu(r)$ and $m(r)$ for a given equation of state. The general integer HSE equation is derived and then the Newtonian limit is taken before fractionalizing. We consider three cases: a fractional HSE equation and integer mass relation, an integer HSE equation and fractional mass relation and both HSE and the mass relation fractionalized. In all three cases, the fractional models can describe objects that are smaller and less massive than those described by integer models.

In the next section we briefly review the integer models. The fractional models are developed in the third part of the paper where we also compare the fractional model details to the standard description of an incompressible fluid star. In Section 4 we develop an effective value for density which allows a non-infinite effective sound speed to be defined. Some possible reasons for the differences between fractional and integer stars are discussed in the last part of the paper. We use the Caputo fractional derivative [16] to develop the models. A brief review of the differences following from using the Caputo, Riemann-Liouville

and Riesz derivatives is in the Appendix.

II. INCOMPRESSIBLE STARS

A. HSE and Mass relation

The HSE condition assumes a perfect fluid stress energy content for the star

$$T^{ij} = (\rho + \frac{P}{c^2})U^iU^j + Pg^{ij}, \quad (2)$$

where P and ρ are the pressure and the density distributions, respectively. U^i is the four velocity and g^{ij} is the metric tensor. Using the zero divergence condition provides the equation

$$(\rho + \frac{P}{c^2})_{;j}U^iU^j + (\rho + \frac{P}{c^2})(U^i_{;j}U^j + U^iU^j_{;j}) + P_{;j}g^{ij} = 0 \quad (3)$$

with spacelike ($h_{ik} = g_{ik} + U_iU_k$) and timelike (U_iU_k) projections, respectively, as

$$(\rho + \frac{P}{c^2})U^k_{;j}U^j + P_{;j}g^{kj} = 0, \quad (4)$$

$$\rho_{;j}U^j + (\rho + \frac{P}{c^2})U^j_{;j} = 0. \quad (5)$$

The field equations for the metric, Eq. (1), provide the relation between the mass $m(r)$ inside a radius r , and the density as

$$\frac{dm(r)}{dr} = 4\pi\rho r^2. \quad (6)$$

Using this, the relativistic hydrostatic equilibrium equation, Eq.(4), is written as

$$\frac{dP}{dr} = -\frac{G(\rho + \frac{P}{c^2})}{r^2}(1 - \frac{2Gm}{rc^2})^{-1}(\frac{4\pi r^3 P}{c^2} + m(r)). \quad (7)$$

The vacuum boundary at $r = R$ is defined as $P(R) = 0$ and the total mass of the star is defined as $M = m(R)$. The mass and pressure gradient at the star's center are both zero.

$$m(0) = 0 \text{ and } \left. \frac{dP}{dr} \right|_{r=0} = 0. \quad (8)$$

In the classical limit, $P \ll \rho c^2$, $2Gm/rc^2 \ll 1$, Eq (7) reduces to the Newtonian hydrostatic equilibrium equation

$$\frac{\partial P}{\partial r} = -G\frac{m(r)}{r^2}\rho, \quad (9)$$

Eqs.(6,9) are the basic stellar structure equations for a spherically symmetric non radiating Newtonian star which are to be solved for a given equation of state.

B. Comparison Model: Incompressible stars

Incompressible stars have constant density, $\rho = \rho_c$. This is the model we use for comparison in fractionalizing the structure equations. For each of the fractional models considered, we compare the mass, pressure and radius of the integer star to the fractional star. The standard constant density parameters are

$$m(r) = \frac{4\pi\rho_c}{3}r^3, \quad (10)$$

$$P(r) = P_c - \frac{2\pi G\rho_c^2}{3}r^2, \quad (11)$$

The pressure is zero at the boundary of the star, $r = R$, and this allows the central pressure to be written in terms of the constant density and radius as

$$P_c = \frac{2\pi G\rho_c^2}{3}R^2 \quad (12)$$

The central pressure will take this value for all models. The rapidity of the rise to the zero pressure surface will vary in the fractional models. The radius of the integer star is

$$R = \left(\frac{3P_c}{2\pi G\rho_c^2} \right)^{1/2}, \quad (13)$$

where the mass of the star is

$$M = 4\pi\rho_c \int_0^R r^2 dr = \frac{4\pi}{3} \left(\frac{3P_c}{2\pi G} \right)^{3/2} \frac{1}{\rho_c^2} \quad (14)$$

and the $M - R$ relation is given as

$$M = \frac{2}{G} \left(\frac{P_c}{\rho_c} \right) R. \quad (15)$$

The quantity in brackets, $\frac{P_c}{\rho_c}$, or M/R , is a measure of the compactness of the star as measured by the Einstein red-shift [17]. The larger the ratio, the more mass can be contained within a given radius.

III. FRACTIONAL STELLAR STRUCTURE EQUATIONS

Fractionalizing the partial derivative in the spherical symmetry mass relation, Eq.(6), and hydrostatic equilibrium, Eq. (9), introduces a unit inconsistency. One way to keep

consistent units while fractionalizing is to express the derivatives in terms of a dimensionless quantity. The derivatives in Eq. (6, 9) can be rewritten using a scale parameter

$$\chi = r/R \quad (16)$$

, where R is the radius of the integer star, Eq. (13). The rescaled equations that will be fractionalized are

$$\frac{dm(\chi)}{d\chi} = 4\pi\rho R^3\chi^2. \quad (17)$$

$$\frac{\partial P}{\partial\chi} = -\frac{G}{R}\frac{m(\chi)}{\chi^2}\rho. \quad (18)$$

The scaled radius of the star, following from $P = 0$, will be denoted by χ_o . The three models we consider are a fractional HSE condition with an integer mass derivative, a fractional mass derivative with an integer HSE condition and both mass and HSE fractionalized. The models we develop will reproduce the integer Newtonian results as the fractional parameters approach their integer limit.

A. Model 1: Fractional HSE and Integer Mass-Density relation

Replacing the partial derivative in Eq. (9) with a Caputo derivative (see Appendix), the fractional generalization of the hydrostatic equilibrium equation is

$$\left(\frac{d^\alpha P_{(\alpha)}}{d\chi^\alpha}\right)_C = -\frac{G}{R}\frac{m_{(\alpha)}(\chi)}{\chi^2}\rho_{(\alpha)}, \quad 0 < \alpha \leq 1. \quad (19)$$

The (α) subscript on the mass and stress identify fractional parameters. The pressure, $P_{(\alpha)}$, the mass, $m_{(\alpha)}$, and the density, $\rho_{(\alpha)}$, have their usual units. Using a dimensionless coordinate for the fractional derivative allows G to have its usual Newtonian units. The Caputo derivative and its Laplace transform are defined, respectively, as

$$\left[\frac{d^q f(t)}{dt^q}\right]_C = \frac{1}{\Gamma(1-q)} \int_0^t \left(\frac{df(\tau)}{d\tau}\right) \frac{d\tau}{(t-\tau)^q}, \quad 0 < q \leq 1, \quad (20)$$

$$\mathcal{L} \left\{ \left[\frac{d^q f(t)}{dt^q}\right]_C \right\} = s^q \tilde{f}(s) - s^{q-1} f(0), \quad 0 < q \leq 1, \quad (21)$$

where $\tilde{f}(s)$ is the Laplace transform of $f(t)$. For the first model we keep the spherical symmetry condition as

$$\frac{dm_{(\alpha)}(\chi)}{d\chi} = 4\pi R^2 \chi^2 \rho_{(\alpha)}. \quad (22)$$

Assuming constant density, ρ_c , we write

$$m_{(\alpha)}(\chi) = 4\pi\chi^3 R^3 \rho_c / 3. \quad (23)$$

With this mass function, the fractional stress gradient is

$$\left(\frac{d^\alpha P_{(\alpha)}}{d\chi^\alpha} \right)_C = -\frac{4\pi G R^2 \rho_c^2}{3} \chi, \quad 0 < \alpha \leq 1. \quad (24)$$

Taking the Laplace transform of the fractional hydrostatic equilibrium equation with the boundary condition (or performing the fractional integral), the pressure is

$$P_{(\alpha)}(\chi) = -\frac{4\pi G R^2 \rho_c^2}{3\Gamma(\alpha + 2)} \chi^{\alpha+1} + P_c, \quad (25)$$

Finding the radius from the surface condition, $P_{(\alpha)}(\chi_{o(\alpha)}) = 0$, we write

$$\chi_{o(\alpha)}^{\alpha+1} = \frac{\Gamma(\alpha + 2)}{2}, \quad (26)$$

which indicates that the fractional object is smaller than its integer counterpart, $\chi_{o(\alpha)}^{\alpha+1} < 1$.

The total mass, M_α , is

$$M_{(\alpha)} = m_{(\alpha)}(\chi_{o(\alpha)}), \quad (27)$$

$$M_{(\alpha)} = M \left(\frac{\Gamma(\alpha + 2)}{2} \right)^{3/(1+\alpha)}. \quad (28)$$

This fractional star has a smaller mass and a smaller radius than the integer star, $M_{(\alpha)} < M$.

Examining Eq.(24) one might assume, with no explicit dependence on the fractional parameter, that the fractional and integer stress gradients were the same. However, even with no explicit fractional dependence, the integer and fractional derivative operations are very different and result in stresses with a strong fractional dependence. For this model, the low alpha fractional pressure ($P_{(\alpha)}(\chi) - P_c$) is larger and increases more rapidly than the integer pressure as χ decreases toward the origin. In Part 4 of the paper, we will show that Model 1 objects with their constant density, can be described as an $n=0$ polytrope. Normally, the $n = 0$ polytropes are models for planet-like objects.

B. Model 2: Integer HSE and Fractional Mass-Density relation

We now consider the case where the scaled stellar structure equations are given as

$$\frac{dP_{(\beta)}}{d\chi} = -G \frac{m_{(\beta)}(\chi)}{R\chi^2} \rho_{(\beta)}, \quad (29)$$

$$\left(\frac{d^\beta m_{(\beta)}}{d\chi^\beta} \right)_C = 4\pi\chi^2 R^3 \rho_{(\beta)}, \quad 0 < \beta \leq 1. \quad (30)$$

Assuming $\rho_{(\beta)} = \rho_c$, we generate the following stellar model:

$$P_{(\beta)}(\chi) = -8\pi G R^2 \rho_c^2 \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+3)\Gamma(\beta+2)} \right] \chi^{1+\beta} + P_c, \quad (31)$$

$$m_{(\beta)}(\chi) = 8\pi R^3 \rho_c \frac{\chi^{2+\beta}}{\Gamma(\beta+3)}, \quad (32)$$

$$\chi_{o(\beta)}^{1+\beta} = \frac{\Gamma(\beta+3)\Gamma(\beta+2)}{12\Gamma(\beta+1)}. \quad (33)$$

The mass of the star is

$$M_{(\beta)} = M \left(\frac{\Gamma(\beta+3)\Gamma(\beta+2)}{12\Gamma(\beta+1)} \right)^{1/(1+\beta)} \frac{\Gamma(\beta+2)}{2\Gamma(\beta+1)}. \quad (34)$$

As in the previous model, this fractional star is smaller and has less mass than the integer star. For this case, with the integer Newtonian HSE condition, the stress gradients vary with the fractional parameter because of the fractional mass-radius relation in this model. The stresses, $P_{(\beta)}(\chi) - P_c$, are smaller for this model than for Model 1 and show a less rapid decrease toward the star's center. As in Model 1, the Model 2 stresses are larger than the integer stress. In the 4th part of the paper, an effective density can be defined for this model.

C. Model 3: Models with both $0 < \alpha \leq 1$ and $0 < \beta \leq 1$

For the most general case with $\rho_{(\alpha,\beta)} = \rho_c$, the stellar structure equations are both fractional and are given as

$$\left(\frac{d^\alpha P_{(\alpha,\beta)}}{d\chi^\alpha} \right)_C = -G \frac{m_{(\alpha,\beta)}(\chi)}{R\chi^2} \rho_c \quad (35)$$

$$\left(\frac{d^\beta m_{(\alpha,\beta)}}{d\chi^\beta} \right)_C = 4\pi \chi^2 R^3 \rho_c, \quad (36)$$

with solution

$$P_{(\alpha,\beta)}(\chi) = -\frac{8\pi G R^2 \rho_c^2 \Gamma(\beta+1)}{\Gamma(\beta+3)\Gamma(\alpha+\beta+1)} \chi^{\alpha+\beta} + P_c, \quad (37)$$

$$m_{(\alpha,\beta)}(\chi) = \frac{8\pi \rho_c R^3}{\Gamma(\beta+3)} \chi^{\beta+2}, \quad (38)$$

$$\chi_{o(\alpha,\beta)} = \left[\frac{\Gamma(\beta+3)\Gamma(\alpha+\beta+1)}{12\Gamma(\beta+1)} \right]^{1/(\alpha+\beta)}, \quad (39)$$

$$M_{(\alpha,\beta)} = \frac{6M}{\Gamma(\beta+3)} \left[\frac{\Gamma(\beta+3)\Gamma(\alpha+\beta+1)}{12\Gamma(\beta+1)} \right]^{\beta+2/(\alpha+\beta)}, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1. \quad (40)$$

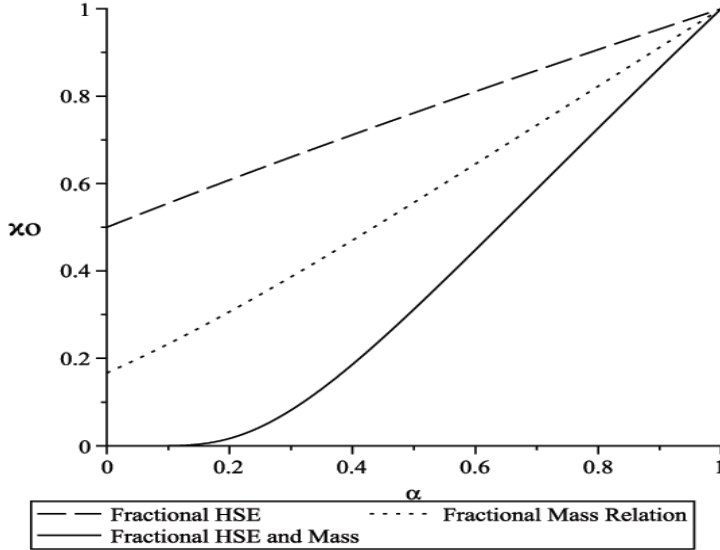


FIG. 1: Scaled surface radius $\chi_o = R_{frac}/R$ vs. α for the three models considered.

For α or β integer, the previous two models are recovered. In the next section we compare the size and mass of all three models.

D. Model Comparison

Models 1 and 2 differ in the size of the stress and stress gradients for the same χ and fractional parameter, with the fractional HSE models having the larger values. A comparison of the three fractional star radii and mass to those of the integer star are shown in Figures (1, 2) for the case of equal fractional index, $\alpha = \beta$. Figure 1 shows the scaled radius versus the fractional index. All three fractional models are smaller than their integer counterpart. For the same α , a fractional hydrostatic equilibrium condition coupled to an integer mass relation will produce a larger star than the fractional star with an integer hydrostatic equilibrium condition.

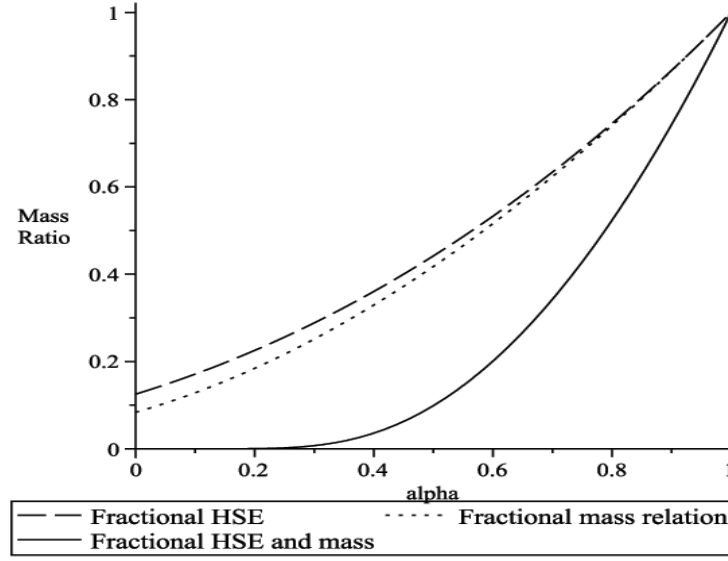


FIG. 2: Fractional Mass/Integer Mass for the three models considered

Figure 2 shows the ratio of the fractional mass models to the integer models. All three fractional models have a smaller mass and radius than their integer counterpart; for $\alpha = \beta$, the fractional HSE condition (Model 1) star has the largest mass, radius and stress of the three models fractional models considered.

Another way of comparing the three models is with an average density. The model results are all scaled in terms of a comparison integer incompressible star. Using the general third model, an average density can be defined

$$\frac{\rho_{av(\alpha,\beta)}}{\rho_c} = \frac{M_{(\alpha,\beta)}}{4\pi\chi_{o(\alpha,\beta)}^3/3} = \frac{6}{\Gamma(\beta+3)} \left[\frac{\Gamma(\beta+3)\Gamma(\alpha+\beta+1)}{12\Gamma(\beta+1)} \right]^{\beta-1/(\alpha+\beta)},$$

where the constant density for the integer stars defined by Eq. (10) has been used as a comparison. For very fractional stars (low values of α and β), the ratio can be very large. For example, for $\alpha = \beta = 0.1$, the ratio is about 6000 while for higher values of the fractional parameters, the ratio can be slightly larger than 1. The comparison object can be an ordinary star or a model astrophysical compact object like a planet, a white dwarf, a

neutron star or a hybrid compact object [18, 19]. One should note that the average density for the model 1 stars ($\beta = 1$), is the same as the integer value so this definition of average density will not explain any Model 1 differences. In the next section we discuss systems equivalent to the fractional models. Using an effective equation of state, we will show that the Model 1 stars are effective $n=0$ polytropes while Model 2 stars have a polytropic index covering a range of fractional values.

IV. EQUIVALENT SYSTEMS

Comparing fractional mass and radii to integer values is one way to show the differences between fractional and integer incompressible stars. A way to understand the differences is to introduce effective values of the fluid parameters, the values the parameters would have if the structure equations were not fractionalized. We begin by defining an effective density and then look at a range of effective gravitational accelerations.

A. Effective Density

An effective density can be found starting with Eq. (30) for constant density and taking another fractional derivative

$$\left(\frac{d^{1-\beta}}{d\chi^{1-\beta}}\right)_C \left[\left(\frac{d^\beta m}{d\chi^\beta}\right)_C\right] = 4\pi R^3 \left(\frac{d^{1-\beta}}{d\chi^{1-\beta}}\right)_C [\chi^2 \rho_c] \quad (41)$$

and using the relation [20]

$$\left(\frac{d^{1-\beta}}{d\chi^{1-\beta}}\right)_C \left[\left(\frac{d^\beta m}{d\chi^\beta}\right)_C\right] = \frac{dm}{d\chi} - \frac{\left(\frac{d^\beta m}{d\chi^\beta}\right)_C \Big|_{\chi=0}}{\Gamma(\beta)\chi^{1-\beta}}, \quad 0 < \beta < 1. \quad (42)$$

Since for $\rho = \rho_c$,

$$\left(\frac{d^\beta m(\chi)}{d\chi^\beta}\right)_C \Big|_{\chi=0} = 0, \quad (43)$$

we can relate the integer mass derivative to the fractional result

$$\frac{dm}{d\chi} = 4\pi R^3 \rho_c \left(\frac{d^{1-\beta}}{d\chi^{1-\beta}}\right)_C [\chi^2] = 4\pi R^3 \rho_c \frac{2\chi^{1+\beta}}{\Gamma(2+\beta)} = 4\pi R r^2 \left[\frac{2\rho_c \chi^{\beta-1}}{\Gamma(\beta+2)}\right], \quad (44)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho_c \frac{2\chi^{\beta-1}}{\Gamma(2+\beta)}. \quad (45)$$

This allows us to define an effective density in terms of the ordinary spherical symmetry condition as

$$\frac{dm}{dr} = 4\pi r^2 \rho_{eff}(\chi), \quad (46)$$

where

$$\rho_{eff} = \left[\frac{2\rho_c \chi^{\beta-1}}{\Gamma(\beta+2)} \right]. \quad (47)$$

This is the equivalent mass density that one would find in terms of the ordinary, non-fractional, spherical symmetry condition. It diverges at the origin but the mass is well behaved. This definition will only apply to Model 2 objects.

B. Effective Equation of State

Using the fractional stress and the effective density, an effective equation of state can be written down and used to place limits on the range of the fractional parameters. Using the pressure relation, Eq. (37), the general fractional radius, Eq. (39) and the effective density we can write an equation of state

$$P_{(\alpha,\beta)}(\chi) - P_c = -P_c \frac{12\Gamma(\beta+1)}{\Gamma(\beta+3)\Gamma(\alpha+\beta+1)} \left(\frac{\Gamma(\beta+2)}{2} \right)^{\frac{\alpha+\beta}{\beta-1}} \left(\frac{\rho_{eff}}{\rho_c} \right)^{\frac{\alpha+\beta}{\beta-1}}. \quad (48)$$

This is in a "polytropic" form relative to pressure gauged to the central value.

$$P_{(\alpha,\beta)}(\chi) - P_c = K \rho_{eff}^{(n+1)/n} \quad (49)$$

with polytropic index

$$n = (1 - \beta)/(1 + \alpha) \quad (50)$$

The models we have described are for an incompressible fluid which is usually an $n = 0$ standard polytrope. Here the gauge polytrope, $n = 0$, corresponds to a fractional HSE condition (Model 1) with the expected constant density across the incompressible fluid and an integer mass relation, $\beta = 1$, $0 < \alpha \leq 1$. The second model, a fractional mass relation, corresponds to a general polytrope with index $n = (1 - \beta)/2$. The $n = 0$ polytrope can be used to describe spherical planet-like objects while the higher values can describe stars, their compactness and their modes of core heat transport. White dwarf models have indices greater than 1. Neutron stars have polytropic indices ranging from about 1/2 to 1. Here, the largest values of n corresponds to very low values of (α, β) and will describe fractional polytropes.

A simple calculation of the speed of sound, V_s , would give $V_s \rightarrow \infty$ for an incompressible fluid. An effective density associated with a fractional object allows a finite effective speed of sound from this equation of state

$$V_s^2 = \frac{\partial P_{(\alpha,\beta)}(\chi)}{\partial \rho_{eff.}} = \frac{P_c}{\rho_c} \frac{6\Gamma(\beta+1)(\alpha+\beta)\Gamma(\beta+2)}{\Gamma(\beta+3)\Gamma(\alpha+\beta+1)(1-\beta)} \chi^{\alpha+1} \leq c^2. \quad (51)$$

The acoustic speed cannot exceed light speed and this provides, at the surface, an inequality on the P_c/ρ_o ratio for Models 2 and 3

$$\frac{P_c}{\rho_c c^2} \leq \frac{2(1-\beta)}{\Gamma(\beta+2)(\alpha+\beta)} \left(\frac{12\Gamma(\beta+1)}{\Gamma(\beta+3)\Gamma(\alpha+\beta+1)} \right)^{(1-\beta)/(\alpha+\beta)}. \quad (52)$$

In considering Newtonian stars, the HSE condition follows from the TOV equation with the condition $P \ll \rho c^2$. Graph 3 shows the maximum $P_c/\rho_c c^2$ ratio versus α for three choices of beta. P_c/ρ_c are input model parameters, with P_c set by its integer value. From Eq. (15), the integer ratio is

$$\frac{P_c}{\rho_c} = \frac{2\pi G \rho_c R^2}{3} = \frac{GM}{2R}.$$

The ratio has a very small upper limit for β and α close to their integer value. This does not preclude choosing smaller values of (α, β) in building fractional models so long as the actual value of P_c/ρ_c is small and under its limit. The Newtonian fractional models will certainly be useful in describing small deviations from the integer values. Olson [21] has discussed equations of state for maximally incompressible neutron stars in the low temperature limit using a stricter V_s/c limit based on stellar stability. A general relativistic treatment could provide a higher limit parameter range for these fractional spherical objects.

C. Effective g-field

An effective gravitational acceleration can also be defined. We use two methods: (1) solving a fractional Poisson equation and (2) from a relation between the fractional and integer derivatives using the effective density. The two methods produce the same result when $\rho_{eff} = \rho_c$.

1. Fractional Poisson Equation

The Poisson equation for the gravitational acceleration is

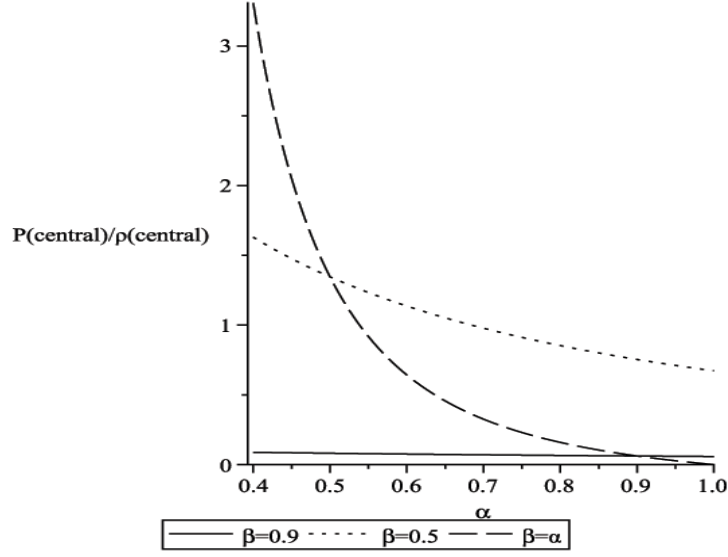


FIG. 3: Central Pressure/Central Density vs. α for three choices of β

$$\frac{dg}{dr} = -4\pi G\rho. \quad (53)$$

The fractional equation with constant density is written as

$$\frac{d^\gamma g}{d\chi^\gamma} = -4\pi GR\rho_c \quad (54)$$

with solution

$$g_{(\gamma)} = \frac{4\pi GR\rho_o\chi^\gamma}{\Gamma(1+\gamma)}.$$

Dividing by the integer value, g , gives

$$4\pi Gr\rho_c \frac{g_{(\gamma)}}{g} = \frac{\chi^{\gamma-1}}{\Gamma(1+\gamma)}. \quad (55)$$

2. The HSE Condition

Fractionalizing the mass, density affects the hydrostatic equilibrium condition and this can be used to get an effective value for the gravitational acceleration. We first write the

ordinary hydrostatic equilibrium equation, Eq. (9), for $\alpha = 1$ as

$$\frac{dP}{dr} = g(r)\rho. \quad (56)$$

Following the same procedure that was used to find an effective density, we relate the regular stress derivative to the fractional Caputo derivative with the relation

$$\frac{dP}{d\chi} = \left(\frac{d^{1-\alpha}}{d\chi^{1-\alpha}} \right)_C \left[\left(\frac{d^\alpha P}{d\chi^\alpha} \right)_C \right] + \frac{\left(\frac{d^\alpha P}{d\chi^\alpha} \right)_C \Big|_{\chi=0}}{\Gamma(\alpha)\chi^{1-\alpha}}, \quad 0 < \alpha \leq 1. \quad (57)$$

For constant density, with the mass boundary condition $m(0) = 0$, Eq. (8), the second term vanishes. Substituting the fractional pressure derivative from Eq. (32) along with the fractional mass relation from Eq.(41) we have into the above equation we find

$$\frac{dP_{(\alpha,\beta)}(\chi)}{d\chi} = \left(\frac{d^{1-\alpha}}{d\chi^{1-\alpha}} \right)_C \left(-\frac{8\pi G\rho_c^2 R^2}{\Gamma(\beta+3)} \chi^\beta \right), \quad (58)$$

$$\frac{dP_{(\alpha,\beta)}}{d\chi} = -\frac{8\pi G\rho_c^2 R^2 \Gamma(\beta+1)}{\Gamma(\beta+3)\Gamma(\beta+\alpha)} \chi^{\alpha+\beta-1}. \quad (59)$$

Rewriting in terms of a regular coordinate derivative this becomes

$$\frac{dP_{(\alpha,\beta)}(\chi)}{dr} = -\frac{8\pi G\rho_c^2 R \Gamma(\beta+1)}{\Gamma(\beta+3)\Gamma(\beta+\alpha)} \chi^{\alpha+\beta-1}. \quad (60)$$

Using the effective density, one could write

$$\frac{dP_{(\alpha,\beta)}(\chi)}{dr} = -g_{eff}(\alpha, \beta) \rho_{eff}(\beta), \quad (61)$$

giving an effective acceleration

$$g_{eff(\alpha,\beta)}(\chi) = 4\pi G\rho_c R \frac{\Gamma(\beta+1)\Gamma(\beta+2)}{\Gamma(\beta+3)\Gamma(\beta+\alpha)} \chi^\alpha. \quad (62)$$

Taking the ratio to the regular value of $g_{(1,1)}(\chi) = 4\pi G\rho_c R \chi$ gives

$$\frac{g_{eff(\alpha,\beta)}(\chi)}{g_{(1,1)}(\chi)} = \frac{3\Gamma(\beta+1)\Gamma(\beta+2)}{\Gamma(\beta+3)\Gamma(\beta+\alpha)} \chi^{\alpha-1}. \quad (63)$$

Note that for $\beta = 1$

$$\frac{g_{eff(\alpha,1)}(\chi)}{g_{(1,1)}(\chi)} = \frac{\chi^{\alpha-1}}{\Gamma(1+\alpha)}, \quad (64)$$

which is the value obtained by directly fractionalizing the Poisson equation with $\gamma = \alpha$. For $r < R$, effective g-field, $g_{(\alpha,1)}$ has a stronger power law dependence than the integer g-value.

$$g_{eff(\alpha,1)}(r) = -\frac{Gm(r)}{R^{\alpha-1}\Gamma(1+\alpha)r^{3-\alpha}}$$

$$\frac{1}{r^2} \rightarrow \frac{1}{r^{3-\alpha}}. \quad (65)$$

The effective gravitational acceleration, along with the effective density, provides some insight into the origins of the Model 1 and Model 2 fractional/integer differences. Eq. (25) gives the stress for the fractional Model 1 star. If an ordinary derivative wrt χ is taken, as expected it is

$$\frac{dP_{(\alpha)}}{dr} = -\rho_c g_{eff(\alpha,1)}(r).$$

This is the standard form for the stress gradients needed to support the stellar material. The larger Model 1 stresses and smaller more massive stars compared to integer models can be motivated by the higher effective g-values. A similar calculation using Eq. (31) for the Model 2 pressure gives a similar equation but involving the effective density for Model 2, Eq. (47),

$$\frac{dP_{(b)}}{dr} = -\rho_{eff} g_{eff(\alpha,1)}(r)$$

While the effective density for the Model 2 stars is larger than the constant density for Model 1 stars, the Model 2 effective gravity is smaller over the low beta ranges than in Model 1. The larger effective gravitational acceleration produces the larger stresses and stress gradients for the Model 1 stars.

V. CONCLUSIONS

We have presented some model objects obeying the fractional stellar structure equations. All of the three models considered produce fractional objects that are smaller and less massive than an integer object. The stresses for the three models can be written in terms of the central pressure and radius as

$$\text{Fractional HSE: } P_{(\alpha)}(\chi) = P_c \left[1 - \left(\frac{\chi}{\chi_{o(\alpha)}} \right)^{1+\alpha} \right] \quad (66)$$

$$\text{Fractional Mass: } P_{(\beta)}(\chi) = P_c \left[1 - \left(\frac{\chi}{\chi_{o(\beta)}} \right)^{1+\beta} \right] \quad (67)$$

$$\text{Both Fractional: } P_{(\alpha,\beta)}(\chi) = P_c \left[1 - \left(\frac{\chi}{\chi_{o(\alpha,\beta)}} \right)^{\alpha+\beta} \right] \quad (68)$$

The discussion of differences between the fractional stars and their integer counterparts has focused on comparisons and equivalent systems. The effective density for the fractional

mass Model 2 stars is larger than for Model 1 (constant ρ_c) while the effective gravitational accelerations for fractional HSE Model 1 stars can be much larger than for Model 2 objects.

Beyond these comparisons, is there an underlying property of the fractional derivative that is driving comparative differences between the fractional and integer stars? Fractional derivatives are non local in that the derivative involves an integral over a spatial or time region. For static stars, there is no time, hence memory effects do not enter. However, while the fractional derivatives of mass and stress are given at a particular $\chi = r/R$, they do spatially sample the entire star. For Models 2 and 3 ($\beta \neq 1$), the average density is an indicator of the sampling differences with very fractional stars having much larger average densities than integer stars. A possible explanation is that the actual mass distributions of fractional Model 2 and 3 objects are very different than in integer object. The fractional objects could simply be more closely packed than in integer objects. A piece of evidence that argue against the more compact mass distribution explanation is the ratio of total mass to surface radius, Eq. (15). This is less than one for Model 2 stars and, if regarded as an indicator of compactness, indicates that the fractional models are less compact than integer models. For example, the ratio for Sirius B is about 150 times larger than our sun. [17]. However, using M/R as an indicator of compactness comes from red-shift arguments which have not been developed for these simple non-radiating fractional models.

The effective density and sound speed cannot be defined for the gauge polytrope, Model 1, which maintains its constant density. Given the difference in the objects usually modeled by gauge and non-gauge polytropic indices, a direct comparison between Model 1 and the other models may be an unphysical comparison. A Model 1 comparison could be to planets which are smaller and less massive than their integer counterparts for a constant density. A constant density planet is a very simple model and the fractional sampling could be used to model actual differences in structure.

The models presented in this paper used the fractional Caputo derivative. There are other fractional derivatives, each extending the fractional model assumptions. One possibility is the fractional Riesz derivative. Models developed using this derivative have an effective gravitational constant, $G_{(\alpha)}$ and a gravitational field very similar to the effective field considered in the previous section. An effective gravitational constant also can appear in theories of dimension D [22, 23]. For example, in D -dimensions, Gauss's law for spherically

symmetric mass distributions can be written

$$g_D(r) = -G_D \frac{m(r)}{r^{D-1}}, \quad (69)$$

with G_D the gravitational constant in D space. This also suggests that for incompressible stars the fractional models are related to incompressible fluids in dimension $D = 4 - \alpha$. Another possibility is the Riemann-Liouville derivative. Its use requires the introduction of fractional boundary conditions $P^{(\alpha-1)}(0)$ and $m^{(\alpha-1)}(0)$, a different extension of the Caputo models. Both possibilities need further investigation.

VI. APPENDIX

Three of the commonly used fractional derivatives are the Caputo derivative, the Riemann-Liouville derivative and the Riesz derivative. The Caputo derivative, used in the models constructed in this paper, and was developed by modifying the Riemann-Liouville derivative. In this Appendix, we briefly describe the three derivatives, beginning with the Caputo derivative.

A. Caputo Fractional Derivative

In the 1960's Caputo introduced a new definition of the fractional derivative [1, 2, 4, 5, 16, 20]

$$\left[\frac{d^q f(t)}{dt^q} \right]_C = \frac{1}{\Gamma(1-q)} \int_0^t \left(\frac{df(\tau)}{d\tau} \right) \frac{d\tau}{(t-\tau)^q}, \quad 0 < q < 1, \quad (70)$$

which was used by him to model dissipation effects in linear viscosity. The two derivatives are related by

$$\left[\frac{d^q f(t)}{dt^q} \right]_C = \left[\frac{d^q f(t)}{dt^q} \right]_{R-L} - \frac{t^{-q} f(0)}{\Gamma(1-q)}, \quad 0 < q < 1. \quad (71)$$

Laplace transforms of the Riemann-Liouville and the Caputo derivative are given as [6]

$$\mathcal{L} \{ {}^{R-L}_0 \mathbf{D}_t^q f(t) \} = s^q \tilde{f}(s) - \sum_{k=0}^{n-1} s^k \left({}^{R-L}_0 \mathbf{D}_t^{q-k-1} f(t) \right) \Big|_{t=0}, \quad n-1 < q \leq n, \quad (72)$$

$$\mathcal{L} \{ {}^C_0 \mathbf{D}_t^q f(t) \} = s^q \tilde{f}(s) - \sum_{k=0}^{n-1} s^{q-k-1} \frac{d^k f(t)}{dt^k} \Big|_{t=0}, \quad n-1 < q \leq n, \quad (73)$$

where ${}_a \mathbf{D}_t^q f(t) \equiv \frac{d^q f}{[d(t-a)]^q}$.

B. Riemann-Liouville Definition of Differintegral:

The basic definition of fractional derivative and integral, that is, differintegral, is the Riemann-Liouville (R-L) definition:

For $q < 0$, the R-L fractional integral is evaluated by using the formula

$$\left[\frac{d^q f}{[d(t-a)]^q} \right] = \frac{1}{\Gamma(-q)} \int_a^t [t-t']^{-q-1} f(t') dt', \quad q < 0. \quad (74)$$

For fractional derivatives, $q \geq 0$, the above integral is divergent, hence the R-L formula is modified as [20]

$$\left[\frac{d^q f}{[d(t-a)]^q} \right] = \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(n-q)} \int_a^t [t-t']^{-(q-n)-1} f(t') dt' \right], \quad q \geq 0, \quad n > q, \quad (75)$$

where the integer n must be chosen as the smallest integer satisfying $(q-n) < 0$.

For $0 < q < 1$ and $a = 0$, the Riemann-Liouville fractional derivative becomes

$$\left[\frac{d^q f(t)}{dt^q} \right]_{R-L} = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^t \frac{f(t') d\tau}{(t-t')^q}, \quad 0 < q < 1. \quad (76)$$

This provides another approach to fractional structure. The Caputo and the Riemann-Liouville derivatives are related by

$$\left[\frac{d^q f(t)}{dt^q} \right]_C = \left[\frac{d^q f(t)}{dt^q} \right]_{R-L} - \frac{t^{-q} f(0)}{\Gamma(1-q)}, \quad 0 < q < 1, \quad (77)$$

The boundary conditions on the mass and pressure are

$$m(0) = 0, \quad P(0) = P_c, \quad (78)$$

Using these the two different fractional derivatives can be related and we have

$$\left[\frac{d^\beta m(r)}{dr^\beta} \right]_{R-L} = \left[\frac{d^\beta m(r)}{dr^\beta} \right]_C, \quad 0 < \alpha < 1, \quad (79)$$

$$\left[\frac{d^\alpha P(r)}{dr^\alpha} \right]_{R-L} = \left[\frac{d^\alpha P(r)}{dr^\alpha} \right]_C - \frac{r^{-\alpha} P_c}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1. \quad (80)$$

Due to the fact that the Laplace transform of the Riemann-Liouville derivative is given as

$$\mathcal{L} \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q \tilde{f}(s) - f^{(q-1)}(0), \quad 0 < q < 1, \quad (81)$$

solving the fractional stellar structure equations in terms of the Riemann-Liouville definition:

$$\left(\frac{d^\alpha P}{dr^\alpha} \right)_{R-L} = -G_\alpha \frac{m(r)}{r^2} \rho, \quad (82)$$

$$\left(\frac{d^\beta m}{dr^\beta} \right)_{R-L} = 4\pi r^2 \rho \quad (83)$$

demands using the boundary conditions in terms of fractional derivatives:

$$P^{(\alpha-1)}(0) \text{ and } m^{(\alpha-1)}(0), \quad (84)$$

C. Riesz Derivative

The Riesz derivative is defined with respect to its Fourier transform [6]

$$\mathcal{F}\{\mathbf{R}_t^q f(t)\} = -|\omega|^q g(\omega), \quad 0 < q < 2, \quad (85)$$

as

$$\mathbf{R}_t^q f(t) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\omega|^q g(\omega) e^{i\omega t} d\omega, \quad (86)$$

where $g(\omega)$ is the Fourier transform of $f(t)$. Note that

$$\mathbf{R}_t^2 f(t) = \frac{d^2}{dt^2} f(t). \quad (87)$$

The Riesz derivative provides another approach to fractional stellar structure. We first write the stellar structure equations as

$$\vec{\nabla} P(\vec{r}) = -\rho \vec{\nabla} \phi(\vec{r}), \quad (88)$$

$$\nabla^2 \phi(\vec{r}) = 4\pi G \rho(\vec{r}), \quad (89)$$

where $\phi(\vec{r})$ is the gravitational potential and fractionalize the gravitational field equation Eq. (85). We now use the fractional generalization of the three dimensional Laplacian in terms of the Riesz derivative as [6]

$$\Delta^{\alpha/2} \phi(\vec{r}) = -\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 \vec{k} \tilde{\phi}(\vec{k}) |\vec{k}|^\alpha e^{i\vec{k} \cdot \vec{r}}, \quad 1 < \alpha \leq 2, \quad (90)$$

where $\tilde{\phi}(\vec{k})$ is the Fourier transform of $\phi(\vec{r})$:

$$\tilde{\phi}(\vec{k}) = \int_{-\infty}^{\infty} d^3 \vec{r} \phi(\vec{r}) e^{-i\vec{k} \cdot \vec{r}}, \quad (91)$$

$$\phi(\vec{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 \vec{k} \tilde{\phi}(\vec{k}) e^{i\vec{k} \cdot \vec{r}}. \quad (92)$$

In other words, the Fourier transform of the fractional Laplacian is

$$\mathcal{F}\{\Delta^{\alpha/2} \phi(\vec{r})\} = -\tilde{\phi}(\vec{k}) |\vec{k}|^\alpha. \quad (93)$$

We now consider the fractional generalization of the stellar structure equations as

$$\vec{\nabla} P(\vec{r}) = -\rho \vec{\nabla} \phi(\vec{r}), \quad (94)$$

$$\Delta^{\alpha/2} \phi(\vec{r}) = 4\pi G \rho(\vec{r}). \quad (95)$$

Taking the Fourier transform of the fractional gravitational field equation, we write the solution as

$$\phi(\vec{r}) = -\frac{4\pi}{(2\pi)^3} G_\alpha \int_{-\infty}^{\infty} d^3 \vec{k} \frac{\tilde{\rho}(\vec{k})}{|\vec{k}|^\alpha} e^{i\vec{k} \cdot \vec{r}}, \quad (96)$$

where $\tilde{\rho}(\vec{k})$ is the Fourier transform of the density distribution. We can also write the above equation as

$$\phi(\vec{r}) = -\frac{4\pi G_\alpha}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 \vec{r}' \int_{-\infty}^{\infty} d^3 \vec{k} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{|\vec{k}|^\alpha} \rho(\vec{r}'), \quad (97)$$

or as

$$\phi(\vec{r}) = -\frac{4\pi G_\alpha}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 \vec{r}' I(\vec{r} - \vec{r}') \rho(\vec{r}'), \quad (98)$$

where

$$I(\vec{r} - \vec{r}') = \int_{-\infty}^{\infty} d^3 \vec{k} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{|\vec{k}|^\alpha}. \quad (99)$$

Using the substitution $k' = k |\vec{r} - \vec{r}'|$, we evaluate the angular part of the above integral to write

$$I(\vec{r} - \vec{r}') = \frac{(2\pi/i)}{|\vec{r} - \vec{r}'|^{3-\alpha}} \left[\int_{-\infty}^{\infty} dk' \frac{k' e^{ik'}}{|k'|^\alpha} \right]. \quad (100)$$

Thus,

$$\phi(\vec{r}) = -\frac{4\pi}{(2\pi)^3} \frac{2\pi}{i} G_\alpha \int_{-\infty}^{\infty} d^3 \vec{r}' \left[\int_{-\infty}^{\infty} dk' \frac{k' e^{ik'}}{|k'|^\alpha} \right] \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^{3-\alpha}}. \quad (101)$$

When $\alpha = 2$, the k' integral can be evaluated as a Cauchy principal value integral [20] as πi , thus yielding the Newtonian potential:

$$\phi(\vec{r}) = -G \int_{-\infty}^{\infty} d^3 \vec{r}' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}. \quad (102)$$

For $1 < \alpha < 2$ we evaluate the integral as

$$\int_{-\infty}^{\infty} \frac{k e^{ik}}{|k|^\alpha} dk = \int_{-\infty}^0 \frac{k e^{ik}}{|k|^\alpha} dk + \int_0^{\infty} \frac{k e^{ik}}{k^\alpha} dk \quad (103)$$

$$= 2i \int_0^{\infty} k^{2-\alpha} \frac{\sin k}{k} dk \quad (104)$$

$$= 2i * 2^{(1-\alpha)} \sqrt{\pi} \Gamma(3/2 - \alpha/2) / \Gamma(\alpha/2). \quad (105)$$

Thus,

$$\phi(\vec{r}) = -2^{(2-\alpha)} \frac{\Gamma(3/2 - \alpha/2)}{\sqrt{\pi}\Gamma(\alpha/2)} G_\alpha \int_{-\infty}^{\infty} d^3\vec{r}' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^{3-\alpha}}. \quad (106)$$

Refining the gravitational constant G_α as

$$\overline{G}_\alpha = 2^{(2-\alpha)} \frac{\Gamma(3/2 - \alpha/2)}{\sqrt{\pi}\Gamma(\alpha/2)} G_\alpha \quad (107)$$

and dropping bar we finally write

$$\phi(\vec{r}) = - G_\alpha \int_{-\infty}^{\infty} d^3\vec{r}' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^{3-\alpha}}. \quad (108)$$

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